ESTIMATION AND TESTING FOR ARCH AND GARCH
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Modelling the volatilty of the Electrolux stock.
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1. Introduction

This lab deals with ARCH and GARCH models (Autoregressive Conditional Heteroscedasticity and Generalized Autoregressive Generalized Autoregressive Heteroscedastic models). Many economic time series display time varying variance in such a way that the variance process can be modeled as an ARMA process.

Volatility clusters, meaning that the variance appears to be high during certain periods and low in other periods often implies an ARMA process in the variance of the process. If the previous period was characterized as high volatility the present and the near future periods are likely to have a high variance as well. Volatility clusters are typical for financial price and return series, exchange rates and inflation rates. In particular, high frequency observations likely to display volatility clustering that can be modelled by ARCH/GARCH methods. High frequency means here monthly and below monthly observations. It is typically not possible to find ARCH/GARCH in frequencies above monthly observations.

The ARCH /GARCH process is often seen as a way of modeling time varying risk. Thus, instead of identifying the factors that explains time varying risk, the consequences of time varying risk can be picked up and modelled with an ARCH/GARCH process. One example is tests of central bank intervention in foreign exchange markets. The question is often, does intervention help to smooth fluctuations? By linking intervention with the variance of foreign exchange rates it becomes possible to test the effects of intervention of foreign exchange volatility.

ARCH/GARCH is not the only way of approaching observed volatility clusters. An alternative, but less common approach is to view the time varying variance as a sign of a missing variable in the model. In this case so-called stochastic volatility (SV) models approaches the problem with so-called latent variable models techniques.

1.0.1. Modelling Arch in Eviews

In Eviews, under Quick estimation methods, look for Estimation methods. Choose Arch. The menu is self-explanatory. Eviews offer all basic ARCH/GARCH options and a view more (not included in this lab)

2. Modelling Arch processes - the basics

The basic ARCH(q) model has two equations, a conditional mean equation and a conditional variance equation. Both equations must be estimated simultaneously, since the variance is a function of the mean. A simple ARCH(1) with an autoregressive first order mean equation and first order variance equation look as follows,
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\[ y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim D(0, h_t) \]  
\[ h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 \]  

(1)  
(2)

Since the variance represents the second moment of the process, it follows that the two equations constitute a system. In this case the mean is an AR(1) process, and the variance process is also an autoregressive process of the first order. Notice that the distribution of the error term is left to be decided. The presence of ARCH means that the normal distribution is not always the best approximation to use. More general we can write an ARCH(q) process as

\[ y_t = E\{y_t | I_t\} + \varepsilon_t, \text{ The mean process,} \]

where \( \varepsilon_t \sim D(0, h_t) \) and

\[ h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2. \text{ The variance process, ARCH(q), of order } q. \]

Sometimes the residual process is written as \( \varepsilon_t = \nu_t h_t^{-1/2} \), where \( \nu_t | I_{t-1} \) is \( \nu_t \sim N(0, 1) \).

It follows that \( \omega \) should be a positive parameter. However, in estimation it might come out negative. Many programs allows the researcher to restrict \( \omega \) to be positive and greater than zero.

The mean equation estimates the conditional mean of the variable. It is important to get the mean equation correctly specified before estimating the ARCH/GARCH model. The mean equation typically can be modelled as an AR process, AR in combination with other explanatory variables, just as a function of other explanatory variables. It is important to test for no autocorrelation in the residuals before estimating the GARCH process.

The variance equation estimates the variance process as a type of autoregressive process. Both equations form a system that is estimated together using maximum likelihood. The mean equation is important because it is not correctly specified the variance estimate will not be good either. The mean equation describes the expected value of the stochastic process \( \{y_t\} \). The mean equation therefore can be an AR, an ARMAX or a structural econometric model etc.

The ARCH model represents a type of moving average in the variance process, which explains the notation of \( q \) in ARCH(\( q \)). Like for AR and MA model we can use the duality to find a simpler specification by combining the two processes into an ARMA type of process. Thus if the ARCH process gets long, a GARCH process will typically offer a better fit. For a variance process we have the Generalized
AutoRegressive Conditional Heteroscedasticity model of order $q, p$, $GARCH(q, p)$ model\(^1\)

\[ h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}. \]

(3)

If there are no ARCH or GARCH effects the sum of the coefficients should be zero, $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i = 0$. It follows that the variable $\omega$ is the residual variance and $\omega = \sigma^2$. The sum of the coefficients $\sum \alpha_i + \sum \beta_i$ shows the long-run solution of the GARCH process. If the coefficients sum to unity, $\sum \alpha_i + \sum \beta_i = 1$, we talk about an Integrated GARCH (IGARCH) process.

The most typical model in empirical work is the $GARCH(1, 1)$ model. Quite often the coefficients of the $GARCH(1, 1)$ will sum to near unity. In an IGARCH (Integrated GARCH) model the parameters adds to unity. This process is stationary provided that $\omega > 0$. However, the variance can be said to have a unit root and there is no finite unconditional variance and the estimated (forecasted) variance grows linearly over the forecast horizon.

The ARCH and GARCH models are a departure for further modeling. There is a huge number of different types of model that one can model, but before moving to the exotic versions. Make sure that you understand the GARCH model.

We can typically do the following. 1) Try different lags in the ARCH/GARCH process. Second, is the assumption of a normal distribution OK or should we try something else. Typically, the t-distribution is the first alternative. Third, is the volatility also a part of the mean equation. Is there M-GARCH, GARCH in mean? Fourth, try some other options like the EGARCH model (exponential GARCH).

2.1. Estimation practicalities

The estimation of ARCH/GARCH processes is easy to perform given a program. However, it is important to understand that the algorithms behind the maximum likelihood routine that does the estimation is non-trivial. This means that different software, different generation of the same software can give different results in estimation. The advice is to accept this fact, use the factory settings in the program, use a recent version of the program and do not change the factory settings for the MLE unless you are an advanced user.

\(^1\)Notice that some authors and programs might change the order of $p$ and $q$ here. Usually they always mean the same. Thus, $GARCH(q, p) \equiv GARCH(p, q)$, and ARCH(q) is associated with alpha parameters, and ARCH(q, p) where $p$ refers to the order of the lagged $h_t : s$ and the associated beta parameters.
3. What is the Meaning of ARCH/GARCH?

Finding heteroscedasticity is a sign of misspecification of the model. Exactly what is causing this is not possible to say. It could be an erroneous dynamic structure or a missing explanatory variable. In fact if there is a missing explanatory variable that is totally orthogonal to the other explanatory variables in the model the only way of detecting this is through the ARCH process.

In finance applications the ARCH/GARCH process is often understood as the reflection of a time varying risk premium, a process not captured by other risk factors.

The classical linear regression (CLR) model assumes the residual variance is constant over the sample, \( h_t = \sigma_t^2 \). Each residual variable in the CLR is an independent random variable with the same expected mean zero and variance around the mean.

Independence between random variables means independece in terms of moments. This is stronger condition than lack of correlation, which is related to specific moments like the moment of the residual process in a regression. In a regression, the first condition is that the residual should not display autocorrelation over time, so the expected mean of each residual is uncorrelated with previous values of the residual. \( E(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, ..., \varepsilon_0) = E(\varepsilon_t) = \mu_t = 0 \). To have an uncorrelated second moment, the variance of each residual should be free from heteroscedasticity, such that \( E(h_t|h_{t-1}, h_{t-2}, ..., h_0) = E(h_t) = \sigma_t^2 \). Notice the difference between not correlated and the stronger term of independence between two (random variables).

If there is correlation in the variance process, the outcome is a loss of efficiency of the estimates. A loss of efficiency does not affect the parameter estimates otherwise. The estimates are still unbiased and consistent as long as the model is stationary and the model is otherwise correctly specified.

However, a significant ARCH test, or any other significant heteroscedasticity test, is a sign of misspecification. In this situation we might have forgotten an important explanatory variable, we might have misspecified the dynamics of the process. The estimation of an \( ARCH(q) \) model is usually the first step in modeling volatility, in the same way as an AR(p) process is tried to investigate the general autoregressive lag structure.

3.1. The Data in this Lab

The data in this lab is the stock.xls data. There are the stock price on Electrolux, The market index, a risk-free rate (not necessary here) and a sector index for the manufacturing sector (not needed here). From these data form the return on Electrolux and the market as,

\[
r_i = \ln P_t - \ln P_{t-1} = \Delta \ln P_t. \tag{4}
\]
3.2. The First Step- Find the mean and the variance

In this lab start by identifying an appropriate AR(p) process for the return on the Electrolux stock. For this type of series the outcome is usually zero lags or one lag. Next, use the basic settings of the program to find a suitable ARCH/GARCH model. Try different specifications, look for significant parameters and low information criteria.

Further, look at the frequency distribution of the return series and the estimated residual from the mean equation. Do they display fatter tails and kurtosis compared to the normal distribution. Do the ACF and PACF the series display ARMA processes? ARCH/GARCH display similar characteristics to ARMA models, and can in principle be identified through ACF:s and PACF:s.

Consider the following questions:

- Can the GARCH(1, 1) be improved?
- Try different lags, including an ARCH(q), with say lag order 5. Compare the information criteria when selecting models in combination with significant lags of q.
- Try the t-distribution instead. Is it better? (Information criteria and sign of t-value-parameter)
- Investigate the estimated residual. Of course, the model should not test significantly for ARCH after modelling and estimating the GARCH process. It is expected to reject normality. Make sure that the test for autocorrelation does not include a very long lag structure.

3.3. The Second Step - Test for ARCH/GARCH-in-Mean

One possibility is that the time varying volatility is representing some missing process in the mean equation. We can therefore put the ARCH/GARCH process back into the mean equation and test for significance. The GARCH-M, for an AR(1) process, looks like,

\[ y_t = a_0 + a_1 x_{t-1} + \delta X_t + \epsilon_t, \]  

(5)

where \( X \) can be (i) \( \hat{h}_t, \hat{h}_t^{1/2} \) or \( \ln \hat{h}_t \) It is difficult to say a priori if any of these variations will be significant, or which is the better one.

3.4. The Third Step - Test for Exogenous Variables in the GARCH Process

It is possible that the ARCH/GARCH process is driven by other explanatory variables. The GARCH process could look like

\[ h_t = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} + \gamma_1 y_t, \]  

(6)
where \( y_t \) is some explanatory variable that explains the time varying variance process \( h_t \). The process is stationary so we can use standard methods for inference.

In this lab try to out in the market return series in the GARCH process. In PcGive, when you formulate the mean equation, add the market return series. Next, in the Status menu, down at the left in the Formulate window, indicate that the variable should be included in the ARCH/GARCH model only. Click on \( [H : X_t \text{ in } h_{-t}] \) and the variable is restricted to be included only in the process for \( h_t \).

3.5. The Fourth Step - Try EGARCH

The exponential GARCH model allows for different reaction in the volatility from positive and negative shocks. By comparing an individual residual with its mean we get an indication of the sign of shock, positive or negative. EGARCH allows for different values in the GARCH parameters depending on the direction of the shock. Thus, volatility might be bigger after a negative shock than after a positive shock. This behavior is sometimes observed on financial markets.

The EGARCH model is a logarithmic transformation of the model, which has two implications. The first is that the parameters are 'forced' to be positive, as we expect them to be. The log is defined on positive numbers. A 'problem' in estimation is otherwise that the computer does not that you are modelling a variance, which by definition must be positive and have positive parameters. The estimation might lead to 'wrong' parameters in the sense that are negative and imply a negative variance, which is impossible.

The second implication is that the EGARCH allows for an asymmetric effect in the response to negative and positive shocks in \( \varepsilon_t \). The EGARCH comes out as a slightly more complex model than the GARCH\((q,p)\) model,

\[
\log h_t = \omega + \sum_{i=1}^{q} \{ \theta_1 \varepsilon_{t-i} + \theta_2 \left( | \varepsilon_{t-i} | - E \{ | \varepsilon_{t-i} | \} \right) \} + \sum_{q=1}^{p} \beta_q \log h_{t-i}. \tag{7}
\]

In the output there are two parameters for the effect of each \( \varepsilon_t \). In an EGARCH(1,1), the first parameter \((\theta_1 - \theta_2)\) indicates the response to positive shock in \( \varepsilon_{t-1} \) and the second parameter \((\theta_1 - \theta_2)\) associated with the absolute value indicates the effects of negative shock in \( \varepsilon_{t-1} \).

When estimating the EGARCH model normally distributed residuals is the default. (There is, in some programs, an option of using what is called a generalized error distribution (GED) instead, which involves the gamma distribution. The asymmetric GARCH, and threshold GARCH are other versions on this theme.
3.6. Some Background Theory for the Mean Equation in this Lab

Individual asset returns are not determined in isolation. Intuition tells you that it is adviceable to spread your investments among many different assets in portfolios of assets. Furthermore, people allocate their portfolios in markets, meaning that the decisions to save and allocate investments are totally voluntary. We might assume therefore that the established relative asset prices, returns and the over all market portfolio composition represent equilibrium relations reflecting the different risk-return characteristics among the individual assets. The so-called market model represent this thinking. The market model is simply the following OLS regression

\[ r_{i,t} = \alpha_i + \beta_i r_{m,t} + \epsilon_t \]  

(8)

This equation says that the individual return can be determined as a linear function of the return on the market. Applied to historical data the right hand side of the equation simply tracks the left hand variable, and shows from the Beta value how the individual stocks follows the return on the market portfolio. If the return series are stationary, and the process is stable, we can use these estimates as predictions and apply the thinking of the Capital Asset Pricing Model (CAPM),

\[ E(r_{i,t}) - r_f = \beta_i [E(r_{m,t}) - r_f] \]  

(9)

where \( r_f \) is the risk-free rate, which is deterministic by default in this model. \( E(.) \) is the expectations operator, and \( \beta \) is the Beta coefficient defined as \( \beta = \frac{Cov(r_{m},r_{i})}{Var(r_{m})} \), which is the slope in a linear regression like the one above. The CAPM theory tells you that all individual stocks are priced in this way, such that the expected return on the individual asset is a mark-up on the risk free rate with a risk premium given by \( \beta_i [E(r_{m,t}) - r_f] \). According to CAPM this is the only risk factor you need to model. The residual term should therefore be an innovation with respect to an information set of all alternative risk factors, \( E(\epsilon_i|I_t) = 0 \). There should be nothing else to add to this equation, nor should there be any ARCH/GARCH.

3.7. The Fifth Step - Add more Theory to the Mean Equation

Knowing more about what drives a return series in equilibrium, you can now estimate the market model (from above) for the stock, and test for ARCH and GARCH in the market model.

\[ r_{i,t} = \alpha_i + \beta_i r_{m,t} + \epsilon_t \]  

(10)

\[ \epsilon_t \sim D(0, h_t) \]  

(11)
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You can also ask, is there any GARCH in MEAN in the market model? Can we from the GARCH/ARCH modelling conclude that there is something missing in the market model?

3.8. What to do in this lab?

Go through each of the five steps above. For each step find the best fitting equation, and explain what the equations are telling you. Perform and comment on the various tests suggested in steps 2 to five.

Hand in before Jan 19th, 2011, at 17:00.